Math Camp II

Basics

Yiqing Xu

MIT

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Plan

- Lectures
  1. Preliminaries: notations, function; Linear algebra: vectors, vector space
  2. Linear algebra: matrix algebra; Application: OLS mechanics
  3. Calculus: differentiation; Application: OLS Asymptotics
  4. Calculus: integration
  5. Calculus: integration; R Session
- R Session:
  1. Basic plots
  2. Basic parallel computing
  3. Accessing remote servers
- "Homework": Four of them, expected to be finished in class

Types of numbers

- Natural numbers: $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$
- Integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- Rational numbers: $\mathbb{Q} \equiv \{\frac{a}{b} | a, b \in \mathbb{Z}; b \neq 0\}$
- Real numbers ($\mathbb{R}$)
- Complex numbers ($\mathbb{C}$)

Symbols

Symbol	Explanation
$\neg$	logical negation statement
$\in$	is an element of
$\therefore$	therefore
$\because$	because
$\Rightarrow$	logical "then" statement
$\Leftrightarrow$	if and only iff, also abbreviated "iff"
$\exists$	there exists
$\forall$	for all
$\equiv$	defined as or equivalent to

$\forall x \in \mathbb{Z}^+$ and $x \neg$prime, $\exists y \in \mathbb{Z}^+$ s.t. $x/y \in \mathbb{Z}^+$

Basic Notations

- summation operator and mean: $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$
- product operator: $\prod_{i=1}^{5} X_i = X_1 \cdot X_2 \cdot X_3 \cdot X_4 \cdot X_5 \cdot$
- Set: $\emptyset$, $\cup$, $\cap$, $\setminus$, $\subset$ ($\subseteq$), $A^c (or \bar{A})$, $\in$
- Symbols (1) $\propto$, $\perp$, ∞, $+\infty$, $-\infty$
- Symbols (2) max{}, min{}, argmax$f(x)$, argmin$f(x)$

Pairs of numbers

- Cartesian coordinate system
  - (x, y), for example, $(1, 5)$
  - "x" represents a location on a horizontal **axis**... "y" represents a location on a vertical **axis**
- Straightforward to add additional axes...this increases the **dimensions** of the graph
  - (x, y, z) is a point in 3-Dimensions (3-D)

$\mathbb{R}^1$ and $\mathbb{R}^n$

- $\mathbb{R}^1$ is the set of all **real** numbers extending from $-\infty$ to $+\infty$ — i.e., the real number line.
- $\mathbb{R}^n$ is an *n*-dimensional space (often referred to as Euclidean space), where each of the *n* axes extends from $-\infty$ to $+\infty$.
- Examples:
  1. $\mathbb{R}^1$ is a line.
  2. $\mathbb{R}^2$ is a plane.
  3. $\mathbb{R}^3$ is a 3-D space.
  4. $\mathbb{R}^4$ could be 3-D plus time.
- Points in $\mathbb{R}^n$ are ordered *n*-tuples, where each element of the *n*-tuple represents the coordinate along that dimension.

Interval Notations

- Open interval: $x \in (a, b)$ if $a < x < b$
- Closed interval: $x \in [a, b]$ if $a \leq x \leq b$
- Mixed intervals: For example ∞ is usually an open interval: $x \in [0, \infty)$ if $x \geq 0$.

Open and Closed Sets

Definition

A subset U of the Euclidean n-space $\mathbb{R}^n$ is called open if, given any point x in U, there exists a real number $\epsilon > 0$ such that, given any point y in $\mathbb{R}^n$ whose Euclidean distance from x is smaller than ϵ, y also belongs to U.

Equivalently, a subset U of $\mathbb{R}^n$ is open if every point in U has a neighborhood in $\mathbb{R}^n$ contained in U.

Definition

There are several equivalent definitions of a closed set. Let S be a subset of a metric space. A set S is closed if

1. The complement of S is an open set
2. S is its own set closure
3. S contains all the limit points of sequences in S
4. Every point outside S has a neighborhood disjoint from S

Functions

Definition

A function is a rule that takes a subset of real numbers, $\mathbb{R}$, and assigns to each real number to another real number.

$\mathbb{R}$ is the set of all real numbers (e.g., $-3, -2.1, 0, 5, \frac{1}{3}$) from $-\infty$ to ∞.

More generally, a function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.

$$f : X \to Y$$

Domain and Range/Image

- Some functions are defined only on proper subsets of $\mathbb{R}^n$.
- **Domain**: the set of numbers in X at which $f(x)$ is defined.
- **Range**: elements of Y assigned by $f(x)$ to elements of X, or

$$f(X) = \{y : y = f(x), x \in X\}$$

  Most often used when talking about a function $f : \mathbb{R}^1 \to \mathbb{R}^1$.
- **Image**: same as range, but more often used when talking about a function $f : \mathbb{R}^n \to \mathbb{R}^1$. It can be a subset of the range that is the output of function on a subset of the domain.

Examples of Common Functions

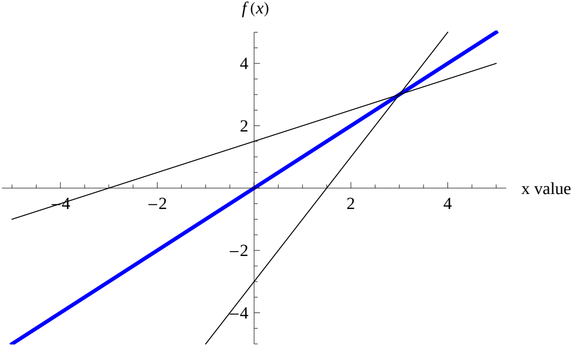
- Linear functions
- Non-Linear functions in polynomial form
- Rational functions: the quotient of two polynomials
- Exponential functions
- Trigonometric functions

Inverse Functions

Definition

Two functions $f(x)$, $g(x)$ are called inverse functions if the following holds for any x:

$$f(g(x)) = x$$



Exponential Functions

Consider the graph of $y = e^x$

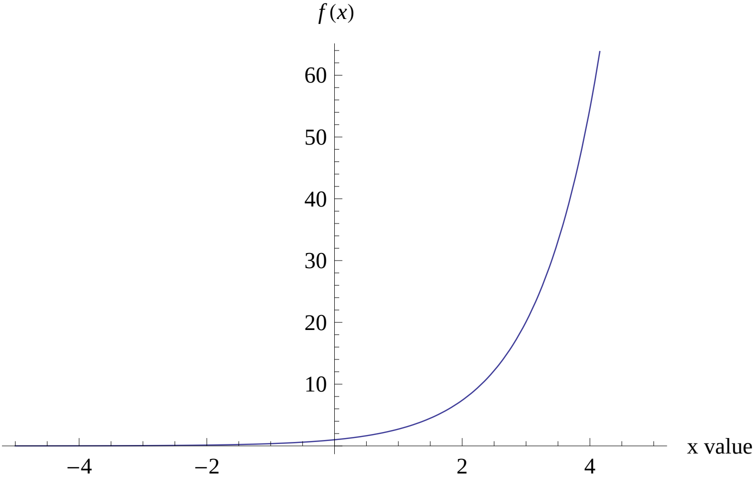


Figure: Graph of $f(x) = e^x$

Properties of Exponential Functions

- strictly positive, never less than or equal to 0
- strictly increasing
- frequently written as $\exp(x)$
- can be expressed as $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}$ (why?)

Gaussian Curve

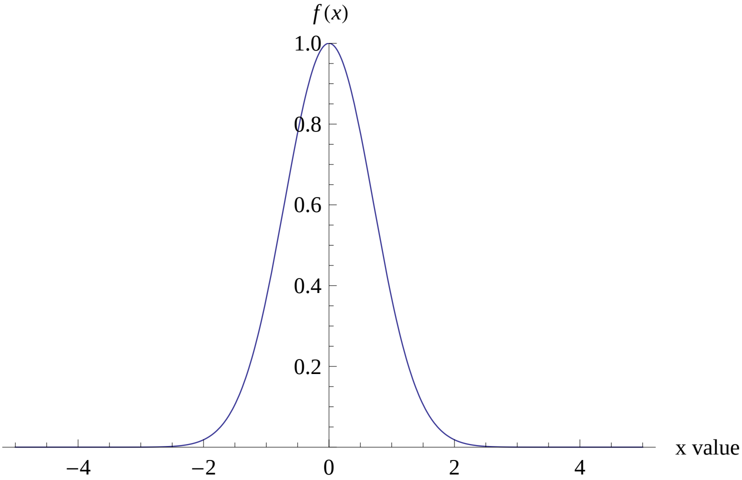


Figure: Graph of $f(x) = e^{-x^2}$

Logarithmic Functions

Definition

$$y = \log_a(x) \ \Leftrightarrow \ a^y = x$$

The logarithmic function gives us the power to which one must raise a to get x. It follows from the definition that:

$$a^{log_a x} = x \text{ and } log_a(a^z) = z$$

Base e logarithms

Definition

The inverse of e^x is called the natural logarithm and is denoted by $ln(x)$. Formally,

$$y = ln(x) \quad \Leftrightarrow \quad e^y = x$$

Rules for Logarithms and Exponentials

1. $a^x a^y = a^{x+y}$
2. $a^{-x} = 1/a^x$
3. $a^x/a^y = a^{x-y}$
4. $(a^x)^y = a^{xy}$
5. $a^0 = 1$

1. $\log(xy) = \log(x) + \log(y)$
2. $\log(1/x) = -\log(x)$
3. $\log(x/y) = \log(x) - \log(y)$
4. $\log(x^y) = y\log(x)$
5. $\log(1) = 0$
6. if $x < y$, then $\log_a x < \log_a y$

Proving Properties of Logs

1. Proof of $\log(xy) = \log(x) + \log(y)$
2. Proof of if $x < y$, then $\log_a x < \log_a y$

The function $\ln(x)$

The graph of $\ln x$ is the inverse of e^x, and hence is the reflection across the line $y = x$.

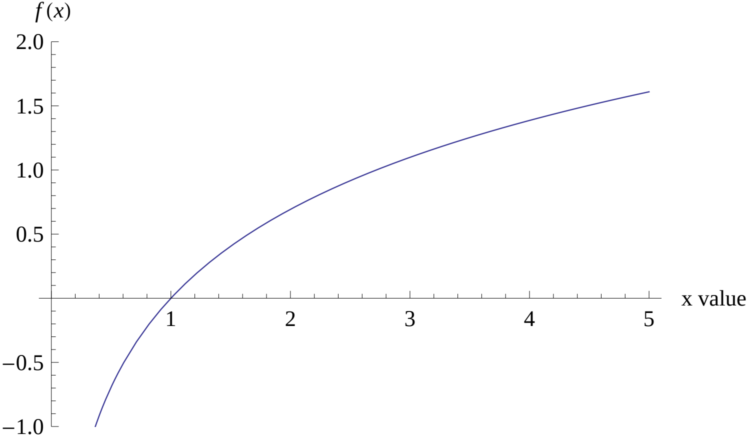


Figure: Graph of $f(x) = ln(x)$

The logistic function

A special function that is frequently used, in a variety of ways, is the function $f(x) = \frac{e^x}{1+e^x}$

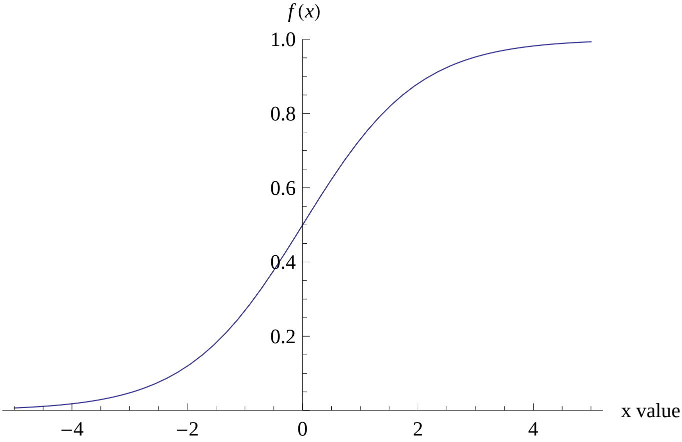


Figure: Graph of $f(x) = \frac{e^x}{1+e^x}$

Composite Functions

Definition

Consider the following two functions. $f : S \to T$ and $g : T \to U$.
The composition $g \circ f : S \to U$ is defined by $(g \circ f)(x) = g(f(x))$.

Note that $(g \circ f)(x) \neq (g \circ f)(x)$

Continuous Functions

Definition

A continuous function is a function $f : X \to Y$ where the pre-image of every open set in Y is open in X.

More concretely, a function f(x) is said to be continuous at point x_0 if

1. $f(x_0)$ is defined, so that x_0 is in the domain of f
2. $lim_{x \to x_0} f(x)$ exists for x in the domain of f
3. $lim_{x \to x_0} f(x) = f(x_0)$

Continuous Functions in $\epsilon - \delta$ Language

A limit c of function $f(x)$ as x approaches a point x_0, $lim_{x \to x_0} f(x) = c$, is defined as:

Definition (Limit)

Given any $\epsilon > 0$, $\exists \delta > 0$ s.t. for $\forall x$ in some domain D and within the neighborhood of x_0 of radius delta (except possibly x_0 itself),

$$|f(x) - c| < \epsilon$$

Definition (Continuous)

If x_0 is in D and

$$lim_{x \to x_0} f(x) = f(x_0) = c,$$

$f(x)$ is said to be continuous at x_0.

Continuous Functions

- If f is differentiable at point x_0, then it is also continuous at x_0
- If two functions f and g are continuous at x_0, then
  1. $f + g$ is continuous at x_0.
  2. $f - g$ is continuous at x_0.
  3. fg is continuous at x_0.
  4. f/g is continuous at x_0 if $g(x_0) \neq 0$.
  5. Providing that f is continuous at $g(x_0)$, $f \circ g$ is continuous at x_0